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LETTER TO THE EDITOR

Intermittency, stochastic growth and phase transition in a simple deterministic partial differential equation with a singular term

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Abstract. We show by numerical integration that the discretized version of simple deterministic partial differential equations with *singular terms* proposed by Zhang exhibit rich spatio-temporal behaviour representing a *mixture of stochastic and deterministic regimes*. Varying the relative weight B of the singular term we have been able to detect transitions in the global behaviour of the solutions by determining their total width $w(t)$. In particular, we have found intermittent solutions as well as a power-law dependence of $W = w(t \to \infty)$ on B.

The emergence of random or random-like behaviour in phenomena formally described by deterministic equations has grown into a much studied field of physics during the last decade. The prototype of such processes is turbulence which, being a fundamental phenomenon in nature, has stimulated extensive theoretical efforts to solve the following beautiful problem: what are the actual mechanisms producing a complex, stochastic spatio-temporal behaviour in a viscous flow beyond a given value of some characteristic parameter? As a simpler case of the original question, one can consider spatially homogeneous systems with complicated temporal dependence described by ordinary differential equations.

For this latter type of stochastic behaviour, much has been understood from the studies of simple systems of nonlinear differential equations and from various simple mappings. The investigation of the so-called chaotic phenomena has lead to a transparent picture of the mechanisms through which deterministic equations may result in a behaviour intermittent or stochastic in time [1]. However, when one is interested in complex dependences in space as well, it is necessary to consider partial differential equations (PDEs) or cellular-automata-type discretized models.

One of the possible strategies for understanding spatio-temporal complexity is to investigate simple model equations resulting in chaotic or turbulent-like patterns [2]. The point is that the Navier–Stokes equations describing the flow of viscous fluids are far too complicated to allow detailed analytic or even numerical studies in the relevant parameter regions. To investigate mechanisms and demonstrate the birth of complicated motion in space (x) and time (t) several nonlinear partial differential equations have been suggested, including the complex Ginzburg–Landau equation [2] or the Kuramoto–Shivashinski (KS) [3–5] equation. It has been argued that the latter equation ($\partial h/\partial t = -\nabla^2 h + |\nabla h|^2 - \nabla^4 h$) describes the propagation of flame fronts. The simultaneous effects of the unstable, nonlinear and the stabilizing terms in the KS equation have been shown to result in chaotic spatio-temporal behaviour of the solutions [6,7], which are fractally rough on large length scales [8–10].

It should be pointed out that in an interesting recent approach to d-dimensional complex spatio-temporal behaviour the various functions associated with these structures are considered as wrinkling (growing rough) surfaces in a $(d+1)$-dimensional space [11]. This development connects the studies of growing fractal surfaces [12–14] to the research of the properties of turbulence-like phenomena since the above-mentioned $(d+1)$-dimensional surfaces can be described in terms of fractal geometry.

From here we conclude that numerical studies of simple deterministic PDEs of the sort used to describe the wrinkling of growing surfaces are expected to be valuable from the point of understanding how stochastic spatio-temporal behaviour emerges in more complex deterministic processes.

In this letter we shall consider perhaps the simplest family of deterministic PDEs producing growing fractal surfaces. These equations, originally proposed by Zhang have the form [15–17]

$$\frac{\partial h(x,t)}{\partial t} = \nabla^2 h(x,t) + \text{singular term} \tag{1}$$

where several forms of the singular term can be used, including

$$|\nabla h|^\alpha \qquad \text{with} \quad \alpha < 1 \tag{2}$$

or

$$\ln(|\nabla h|)\,. \tag{3}$$

In particular, in this letter we shall pay most attention to the equation

$$\frac{\partial h}{\partial t} = \nabla^2 h - B\ln(|\nabla h| + A) \tag{4}$$

where the parameter $A > 0$ is used to control the largest possible value of the singular term, while B *is introduced to monitor the relative strength of the singular term.* In this letter we do not discuss the possible origins of a singular term. Zhang [15] argued that the complex directed-polymer problem leads to an equation analogous to (1). Here we simply assume that the simultaneous effects of more complicated mechanisms can under some conditions be accounted for by a simple singular term of the form we are considering. The time dependence of the roughness W of surfaces generated by (1) with (3) was investigated by Zhang [15] who found that $W \sim t^\beta$ with $\beta \simeq 0.2$. Following Zhang's suggestion, Amar and Family [18] numerically integrated (1) with (2) for $\alpha = \frac{1}{2}$. In this case (2) does not diverge as $\nabla h \to 0$, however, the corresponding term is unstable (in this letter we shall consider singular terms diverging as $\nabla h \to 0$). They determined β and the Lyapunov exponents corresponding to the chaotic behaviour of (1) with (2). They also observed a grooved phase characterized by occasionally occurring and disappearing linear parts embedded in the rough interface.

Our main goal is to demonstrate the various interesting phenomena which are exhibited by (4) *as B is increased from 0.* We shall use the following approach: (i) start with a random or some simple initial surface (no relevant difference has been seen between the two kinds of simulation results), (ii) numerically integrate the equation using a simple discretization scheme (iii) evaluate the data in terms of the surface roughness (the total width) of the advancing surface given by the function $h(t)$. All our simulations are carried out in a $(1 + 1)$-dimensional strip with periodic boundary conditions.

We integrated (4) using the discretization scheme

$$h(x, t + \Delta t) = h(x, t) + \Delta t\{h(x - 1, t) - 2h(x, t) + h(x + 1, t)\} \\ - B\Delta t\{\ln[|(h(x + 1, t) - h(x - 1, t)| + A]\} \qquad (5)$$

where for the integration step Δt in time, in most cases we used 0.05 (the results did not depend on Δt for $\Delta t < 0.1$). Thus, we study the lattice version of (4) which has been shown to be fundamentally different from the continuous one [15, 17]. Since we are considering periodic boundary conditions the solutions must have extrema (or at least one extremum point). In the case of continuous solutions at an extremum point either the term $\ln(|\nabla h|)$ (smooth extremum) or the term $\nabla^2 h$ (sharp kink) diverge. Discretization eliminates these sorts of divergences because starting with random initial conditions the finite difference expression for the gradient takes on the value $\nabla h = 0$ with zero probability while for $\nabla^2 h$ it is always finite even at the sharpest extrema. The width of the strip was typically $L = 512$ grid points, but we have also carried out simulations for $L = 256$ and $L = 1024$ to check whether there is any significant size dependence in our calculations. The parameter A was kept constant and B, the relative weight of the singular term, was increased gradually from zero. The initial condition $h(x, 0)$ was a random surface with heights uniformly distributed between 0.0 and 0.01.

Our findings are demonstrated in figures 1–4. First we present (figures 1(a)–(d)) sets of actual surface configurations for various B to illustrate the qualitative behaviour of the solutions.

(i) Naturally, for $B = 0$ the surface becomes perfectly *smooth* as $t \to \infty$ since, as can be seen from a trivial linear stability analysis, the surface-tension-like term $\nabla^2 h$ leads to the dying out of the perturbations.

(ii) As B becomes larger, at places where ∇h is approximately zero the term $B\ln(|\nabla h| + A)$ is close to $B\ln(A)$ which, for $A \ll 1$, represents a large perturbation to the local velocity of the advancing surface. The strength of this perturbation depends sensitively on how close ∇h is to zero at the given discretization node and this feature, through the nonlinearity of the dependence of the velocity on the local slope, results in the *roughening* of the surface (figure 1(a)).

(iii) For $B > B_L$ the surface becomes piecewise *linear*, consisting of straight line segments of a given slope (figure 1(b)). It is natural that the singular term dominated regime is made of straight line segments which (a) minimize the number of points where $\nabla h = 0$ and, (b) correspond to a trivial steady state because for these segments $\nabla^2 h = 0$ and $\nabla h =$ constant.

(iv) Perhaps most interestingly, the crossover from stochastic to piecewise behaviour is accompanied by a phenomenon analogous to *intermittency*: periods of almost perfectly regular (piecewise) growth regimes are interrupted with intervals of stochastic growth (figure 1(c)). As an intermediate regime we can also observe surface evolution during which parts of the surface become piecewise linear and turn random at later stages while the rest of the surface remains disordered (figure 1(d)).

In order to describe the above changes in the spatio-temporal behaviour in a more quantitative manner we calculate the total width of the surfaces $w^2(t) = \langle h^2 \rangle - \langle h \rangle^2$, where the averaging is made over the $h(x)$ values for $x = 1, \ldots L$ at time t. The *intermittent*

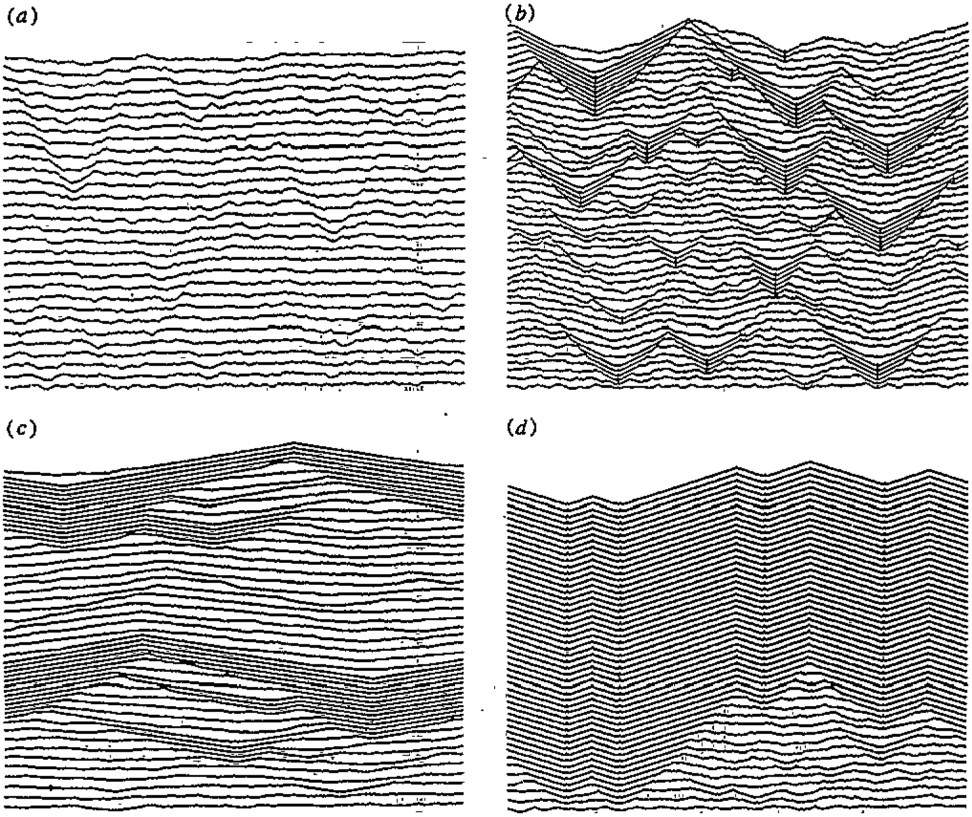

Figure 1. Subsequent 'snapshots' of the evolving surface obtained by numerically integrating (4) for (*a*) $L = 512$, $A = 0.0002$ and $B = 0.002$; (*b*) $B = 0.0053$; (*c*) $B = 0.0061$ and (*d*) $B = 0.01$. All surfaces have been shifted by an amount $-C(B)t$ (this is equivalent to including an extra, irrelevant term $-C$ into the RHS of (4)) in order to show many surfaces (otherwise separated by a much larger gap) in the available area of a figure. In addition, the solutions are 'stretched' in the vertical direction (multiplied by a factor, depending on B, in the range of 200–2000 to enhance the details).

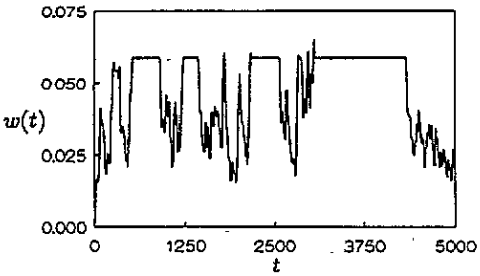

Figure 2. The total width of the growing surfaces $w(t)$ as a function of time for $A = 0.002$ and $B = 0.0061$. The intermittent nature of the solution is demonstrated by the periods of steady-state regime (relatively large, constant value of w corresponding to a piecewise linear solution existing for some time) interrupted by stochastically fluctuating time dependence.

nature of the solution of (4) for $B = 0.0061$ is demonstrated in figure 2. Intervals of the steady-state regime (relatively large constant value of w corresponding to a piecewise linear solution existing for some time) are interrupted by stochastically changing behaviour.

Figure 3(*a*) shows how $W = w(t \to \infty)$ depends on the *relative weight B of the nonlinear term* for $A = 0.0001$. In this plot the $\log(W)$ values approximately follow two

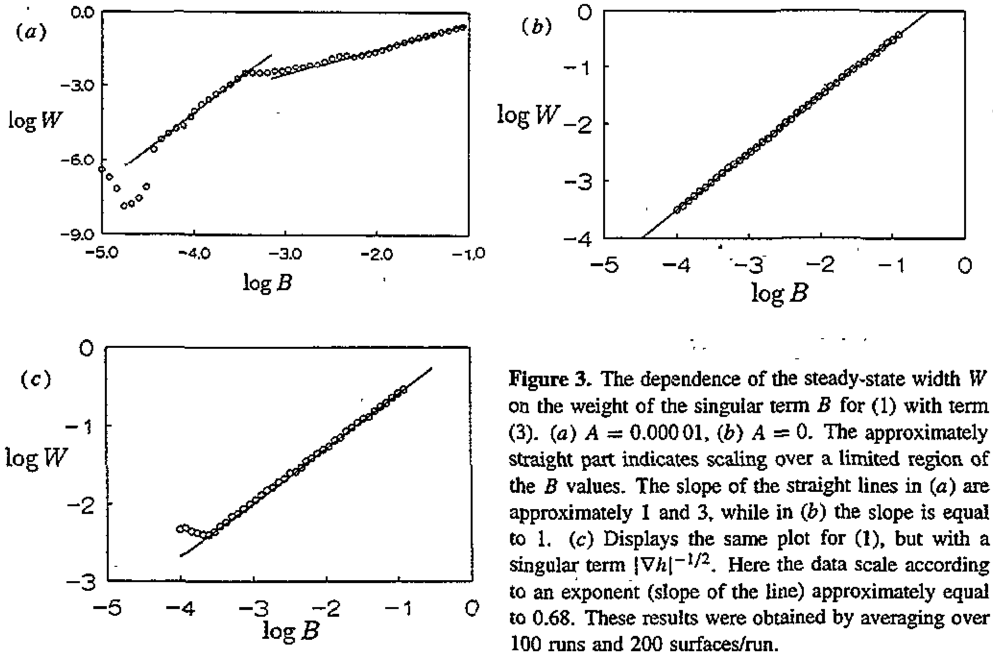

Figure 3. The dependence of the steady-state width W on the weight of the singular term B for (1) with term (3). (a) $A = 0.000\,01$, (b) $A = 0$. The approximately straight part indicates scaling over a limited region of the B values. The slope of the straight lines in (a) are approximately 1 and 3, while in (b) the slope is equal to 1. (c) Displays the same plot for (1), but with a singular term $|\nabla h|^{-1/2}$. Here the data scale according to an exponent (slope of the line) approximately equal to 0.68. These results were obtained by averaging over 100 runs and 200 surfaces/run.

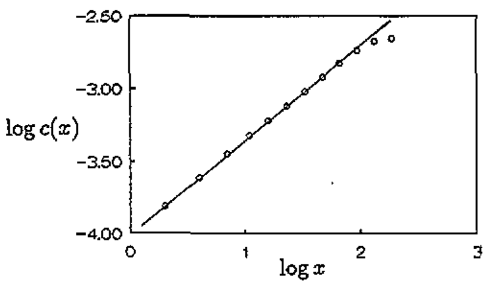

Figure 4. The height–height correlation function $c(x)$ for $A = 0.0002$ and $B = 0.002$. The self-Affine fractal nature of the growing surface is indicated by the straight part in the plot. The slope corresponds to a roughness exponent $H \approx 0.7$.

straight lines as a function of $\log(B)$ indicating a power-law-like dependence of the total surface width in a certain range of the parameter B. For small B the slope is about 3, while for larger B the slope is close to 1. As discussed above the discretized version of the term $\ln(|\nabla h|)$ never becomes equal to infinity; in the simulations the role of A is to introduce the largest possible value for the nonlinear term. Thus, the size of the scaling region and the value of B at which the crossover takes place depend on the value of A. Figure 3(b) shows the results for $A = 0$. In this case the data fall onto a straight line over many decades because the term $\ln(|\nabla h|)$ can assume much larger (but still finite) values than $\ln(|\nabla h + A|)$ for $A > 0$. Figure 3(c) displays the total width versus B for (1) with a singular term $B|\nabla h + A|^{-1/2}$ demonstrating that the behaviour of W is essentially the same independently of the actual form of the singular term. The slope corresponding to the surprisingly straight set of data is ≈ 0.68. The extension of this behaviour depends on the actual value of A. Thus, B plays the role of a control parameter and the change in the behaviour can be interpreted in terms of a morphological phase transition of the rough surface [12, 14]. The value of the parameter A has the simple effect of a scaling cut-off for smaller B. As $A \to 0$ the almost perfect scaling behaviour extends over many orders of magnitude.

Finally, we have investigated the geometry of the surface by calculating the height–height correlation function (for some t) $c(x) = \langle |h(x'+x) - h(x')| \rangle_{x'}$ [12–14]. The fractal roughness of the growing surface for $B = 0.002$ is indicated by the straight part in the plot of $\log c(x)$ as a function of x (figure 4). The corresponding roughness exponent is $H \approx 0.7$, where H is defined by the expression $c(x) \sim x^H$.

In conclusion, we have shown that the discretized version of simple deterministic partial differential equations with singular terms (Zhang equations) exhibit a behaviour which is an interesting *mixture of stochastic and deterministic regimes*. Varying the relative strength B of the singular term we have been able to detect transitions in the global behaviour of the solutions in analogy with some viscous flows in which changes from laminar to intermittent and turbulent régimes take place as the Reynolds number is increased. In our case the emergence of the new type of solution depends on B as a power law with a well defined exponent. The piecewise linear solution we find numerically may be related to the so-called groove instability observed in several surface growth models.

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